

CLASSIFYING HOMOGENEOUS CELLULAR ORDINAL BALLEANS UP TO COARSE EQUIVALENCE

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ABSTRACT. For every ballean X we introduce two cardinal characteristics $\text{cov}^b(X)$ and $\text{cov}^\sharp(X)$ describing the capacity of balls in X . We observe that these cardinal characteristics are invariant under coarse equivalence and prove that two cellular ordinal ballean X, Y are coarsely equivalent if $\text{cof}(X) = \text{cof}(Y)$ and $\text{cov}^b(X) = \text{cov}^\sharp(X) = \text{cov}^b(Y) = \text{cov}^\sharp(Y)$. This result implies that a cellular ordinal ballean X is homogeneous if and only if $\text{cov}^b(X) = \text{cov}^\sharp(X)$. Moreover, two homogeneous cellular ordinal ballean X, Y are coarsely equivalent if and only if $\text{cof}(X) = \text{cof}(Y)$ and $\text{cov}^\sharp(X) = \text{cov}^\sharp(Y)$ if and only if each of these ballean coarsely embeds into the other ballean. This means that the coarse structure of a homogeneous cellular ordinal ballean X is fully determined by the values of the cardinals $\text{cof}(X)$ and $\text{cov}^\sharp(X)$. For every limit ordinal γ we shall define a ballean $2^{<\gamma}$ (called the Cantor macro-cube), which in the class of cellular ordinal ballean of cofinality $\text{cf}(\gamma)$ plays a role analogous to the role of the Cantor cube 2^κ in the class of zero-dimensional compact Hausdorff spaces. We shall also present a characterization of ballean which are coarsely equivalent to $2^{<\gamma}$. This characterization can be considered as an asymptotic analogue of Brouwer's characterization of the Cantor cube 2^ω .

INTRODUCTION

In this paper we study the structure of ordinal ballean, i.e., ballean that have well-ordered base of their coarse structure. Such ballean were introduced by Protasov in [10]. Some basic facts about ordinal ballean are discussed in Section 1. The main result of the paper is presented in Section 2 containing a criterion for recognizing coarsely equivalent cellular ordinal ballean. In Section 3 we shall use this criterion to classify homogeneous cellular ordinal ballean up to coarse equivalence. In Section 4 we apply this criterion to characterize ballean $2^{<\gamma}$ (called Cantor macro-cubes), which are universal objects in the class of cellular ordinal ballean. In Section 4 also we identify the natural coarse structure on additively indecomposable ordinals.

1. ORDINAL BALLEANS

The notion of a ballean was introduced by Protasov [11] as a large scale counterpart of a uniform space and is a modification of the notion of a coarse space introduced by Roe [14]. Both notions are defined as sets endowed with certain families of entourages.

By an *entourage* on a set X we understand any reflexive symmetric relation $\varepsilon \subset X \times X$. This means that ε contains the diagonal $\Delta_X = \{(x, y) \in X \times X : x = y\}$ of $X \times X$ and is symmetric in the sense that $\varepsilon = \varepsilon^{-1}$ where $\varepsilon^{-1} = \{(y, x) \in X \times X : (x, y) \in \varepsilon\}$. An entourage $\varepsilon \subset X \times X$ will be called *cellular* if it is transitive, i.e., it is an equivalence relation on X .

Each entourage $\varepsilon \subset X \times X$ determines a cover $\{B(x, \varepsilon) : x \in X\}$ of X by ε -balls $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$. It follows that $\varepsilon = \bigcup_{x \in X} \{x\} \times B(x, \varepsilon) = \bigcup_{x \in X} B(x, \varepsilon) \times \{x\}$, so the entourage ε can be fully recovered from the system of balls $\{B(x, \varepsilon) : x \in X\}$. For a subset $A \subset X$ we let $B(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon)$ denote the ε -neighborhood of A .

A *ballean* is a pair (X, \mathcal{E}_X) consisting of a set X and a family \mathcal{E}_X of entourages on X (called the *set of radii*) such that $\bigcup \mathcal{E}_X = X \times X$ and for any entourages $\varepsilon, \delta \in \mathcal{E}_X$ their composition

$$\varepsilon \circ \delta = \{(x, z) \in X \times X : \exists y \in X (x, y) \in \varepsilon, (y, z) \in \delta\}$$

is contained in some entourage $\eta \in \mathcal{E}_X$. A ballean (X, \mathcal{E}_X) is called a *coarse space* if the family \mathcal{E}_X is closed under taking subentourages, i.e., for any $\varepsilon \in \mathcal{E}_X$ any entourage $\delta \subset \varepsilon$ belongs to \mathcal{E}_X . In this case the set of radii \mathcal{E}_X is called a *coarse structure* on X . Each set of radii \mathcal{E}_X can be completed to the coarse structure $\downarrow \mathcal{E}_X$ consisting of all possible subentourages $\delta \subset \varepsilon \in \mathcal{E}_X$. In this case \mathcal{E}_X is called a *base* of the coarse structure $\downarrow \mathcal{E}_X$. So, ballean can be considered as coarse spaces with a fixed base of their coarse structure. Coarse spaces and coarse structures were introduced by Roe [14].

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Each subset $A \subset X$ of a ballean (X, \mathcal{E}_X) carries the induced ballean structure $\mathcal{E}_A = \{\varepsilon \cap A^2 : \varepsilon \in \mathcal{E}_X\}$. The ballean (A, \mathcal{E}_A) will be called a *subballean* of (X, \mathcal{E}_X) .

Now we consider some examples of ballians.

Example 1.1. Each metric space (X, d) carries a canonical ballean structure $\mathcal{E}_X = \{\Delta_\varepsilon\}_{\varepsilon \in \mathbb{R}_+}$ consisting of the entourages

$$\Delta_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}$$

parametrized by the set $\mathbb{R}_+ = [0, \infty)$ of non-negative real numbers. The ballean structure $\mathcal{E}_X = \{\Delta_\varepsilon\}_{\varepsilon \in \mathbb{R}_+}$ generates the coarse structure $\downarrow \mathcal{E}_X$ consisting of all subentourages of the entourages Δ_ε , $\varepsilon \in \mathbb{R}_+$.

A ballean X is called *metrizable* if its coarse structure is generated by a suitable metric. Metrizable ballians belong to the class of ordinal ballians. A ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is defined to be *ordinal* if its coarse structure $\downarrow \mathcal{E}_X$ has a well-ordered base $\mathcal{B} \subset \mathcal{E}_X$. The latter means that \mathcal{B} can be enumerated as $\{B_\alpha\}_{\alpha < \kappa}$ for some ordinal κ such that $B_\alpha \subset B_\beta$ for all ordinals $\alpha < \beta < \kappa$. Passing to a cofinal subset of κ , we can always assume that κ is a regular cardinal, equal of the cofinality $\text{cof}(\mathbf{X})$. By definition, the *cofinality* $\text{cof}(\mathbf{X})$ of a ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is equal to the smallest cardinality of a base of the coarse structure $\downarrow \mathcal{E}_X$.

Ordinal ballians can be characterized as ballians $\mathbf{X} = (X, \mathcal{E}_X)$ whose cofinality equals the additivity number

$$\text{add}(\mathbf{X}) = \min \{|\mathcal{A}| : \mathcal{A} \subset \downarrow \mathcal{E}_X, \bigcup \mathcal{A} \notin \downarrow \mathcal{E}_X \setminus \{X \times X\}\}.$$

Proposition 1.2. *A ballean X is ordinal if and only if $\text{cof}(X) = \text{add}(X)$.*

Proof. Assuming that a ballean (X, \mathcal{E}_X) is ordinal, fix a well-ordered base $\{\varepsilon_\alpha\}_{\alpha < \kappa}$ of the coarse structure $\downarrow \mathcal{E}_X$ of \mathcal{E}_X . Passing to a cofinal subsequence, we can assume that $\kappa = \text{cf}(\kappa)$ is a regular cardinal. If $\kappa = 1$, then the ballean (X, \mathcal{E}_X) is bounded and hence for the entourage $X \times X \in \downarrow \mathcal{E}_X$ the family $\mathcal{A} = \{X \times X\}$ has cardinality $|\mathcal{A}| = 1$ and $\bigcup \mathcal{A} = X \times X \notin \mathcal{E}_X \setminus \{X \times X\}$. Therefore, $\text{add}(X) = 1 = \text{cof}(X)$. So, we assume that the regular cardinal κ is infinite and hence $\varepsilon_\alpha \neq X \times X$ for all $\alpha < \kappa$. Since $\text{add}(X) \leq \text{cof}(X)$, it suffices to check that $\text{cof}(X) \leq \text{add}(X)$. The definition of the cardinal $\text{cof}(X)$ implies that $\text{cof}(X) \leq \kappa$. The inequality $\text{add}(X) \geq \kappa \geq \text{cof}(X)$ will follow as soon as we check that for any family $\mathcal{A} \subset \mathcal{E}_X$ of cardinality $|\mathcal{A}| < \kappa$ we get $\bigcup \mathcal{A} \in \downarrow \mathcal{E}_X \setminus \{X \times X\}$. For every set $A \in \mathcal{A}$ find an ordinal $\alpha_A < \kappa$ such that $A \subset \varepsilon_{\alpha_A}$. By the regularity of the cardinal κ , the cardinal $\beta = \sup\{\alpha_A : A \in \mathcal{A}\}$ is strictly smaller than κ . Consequently, $A \subset \varepsilon_{\alpha_A} \subset \varepsilon_\beta$ for every $A \in \mathcal{A}$ and hence $\bigcup \mathcal{A} \subset \varepsilon_\beta$ and $\bigcup \mathcal{A} \in \downarrow \mathcal{E}_X \setminus \{X \times X\}$. This completes the proof of the equality $\text{add}(X) = \text{cof}(X)$ for ordinal ballians.

Now we shall prove that a ballean (X, \mathcal{E}_X) is ordinal if $\text{add}(X) = \text{cof}(X)$. Fix any base $\{\varepsilon_\alpha\}_{\alpha < \text{cof}(X)} \subset \downarrow \mathcal{E}_X$ of the coarse structure $\downarrow \mathcal{E}_X$ of X . By definition of the additivity number $\text{add}(X)$, for every $\alpha < \text{cof}(X) = \text{add}(X)$, the union $\tilde{\varepsilon}_\alpha = \bigcup_{\beta \leq \alpha} \varepsilon_\beta$ belongs to the coarse structure $\downarrow \mathcal{E}_X$. Then $(\tilde{\varepsilon}_\alpha)_{\alpha < \text{cof}(X)}$ is a well-ordered base of the coarse structure $\downarrow \mathcal{E}_X$, which means that the ballean (X, \mathcal{E}_X) is ordinal. \square

An important property of ordinal ballians of uncountable cofinality is their cellularity. A ballean (X, \mathcal{E}_X) is called *cellular* if its coarse structure $\downarrow \mathcal{E}_X$ has a base consisting of cellular entourages (i.e., equivalence relations). It can be shown that a ballean (X, \mathcal{E}_X) is cellular if and only if for every $\varepsilon \in \mathcal{E}_X$ the cellular entourage $\varepsilon^{<\omega} = \bigcup_{n \in \omega} \varepsilon^n$ belongs to the coarse structure $\downarrow \mathcal{E}_X$. Here $\varepsilon^0 = \Delta_X$ and $\varepsilon^{n+1} = \varepsilon^n \circ \varepsilon$ for all $n \in \omega$. This characterization implies the following simple fact.

Proposition 1.3. *Each ordinal ballean $\mathbf{X} = (X, \mathcal{E}_X)$ with uncountable cofinality $\text{cof}(\mathbf{X})$ is cellular.*

Remark 1.4. By [13, Theorem 3.1.3], a ballean X is cellular if and only if it has asymptotic dimension $\text{asdim}(X) = 0$. A metrizable ballean X is cellular if and only if its coarse structure is generated by an ultrametric (i.e., a metric d satisfying the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all point $x, y, z \in X$). More information on cellular ballians can be found in [13, Chapter 3]. For information on space of asymptotic dimension zero, see [4].

Example 1.5. Every infinite cardinal κ carries a natural ballean structure $\mathcal{E}_\kappa = \{\varepsilon_\alpha\}_{\alpha < \kappa}$ consisting of the entourages

$$\varepsilon_\alpha = \{(x, y) \in \kappa \times \kappa : x \leq y + \alpha, \ y \leq x + \alpha\}$$

parametrized by ordinals $\alpha < \kappa$. The obtained ordinal ballean $(\kappa, \mathcal{E}_\kappa)$ will be denoted by $\overleftrightarrow{\kappa}$. Cardinal ballians $\overleftrightarrow{\kappa}$ were introduced in [8]. By Theorem 3 of [8], the ballean $\overleftrightarrow{\kappa}$ is cellular for any uncountable cardinal κ .

Example 1.6. Given an ordinal γ and a transfinite sequence $(\kappa_\alpha)_{\alpha < \gamma}$ of non-zero cardinals, consider the ballean

$$\prod_{\alpha \in \gamma} \kappa_\alpha = \{(x_\alpha)_{\alpha \in \gamma} \in \prod_{\alpha \in \gamma} \kappa_\alpha : |\{\alpha \in \gamma : x_\alpha \neq 0\}| < \omega\}$$

endowed with the ballean structure $\{\varepsilon_\beta\}_{\beta < \gamma}$ consisting of the entourages

$$\varepsilon_\beta = \left\{ ((x_\alpha)_{\alpha \in \gamma}, (y_\alpha)_{\alpha \in \gamma}) \in \left(\prod_{\alpha < \gamma} \kappa_\alpha \right)^2 : \forall \alpha > \beta \ (x_\alpha = y_\alpha) \right\} \text{ for } \beta < \gamma.$$

The ballean $\prod_{\alpha \in \gamma} \kappa_\alpha$ is called the *asymptotic product* of cardinals κ_α , $\alpha \in \gamma$. It is a cellular ordinal ballean whose cofinality equals $\text{cf}(\gamma)$, the cofinality of the ordinal γ .

If all cardinals κ_α , $\alpha \in \gamma$, are equal to a fixed cardinal κ , then the asymptotic product $\prod_{\alpha \in \gamma} \kappa_\alpha$ will be denoted by $\kappa^{<\gamma}$. For a limit ordinal γ the ballean $2^{<\gamma}$ is called a *Cantor macro-cube*. The Cantor macro-cube $2^{<\omega}$ was characterized in [3]. This characterization will be extended to all Cantor macro-cubes in Theorem 4.3.

Balleans are objects of the (coarse) category whose morphisms are coarse maps. A map $f: X \rightarrow Y$ between two balleans (X, \mathcal{E}_X) and (Y, \mathcal{E}_Y) is called *coarse* if for each $\varepsilon \in \mathcal{E}_X$ there is $\delta \in \mathcal{E}_Y$ such that $\{(f(x), f(y)) : (x, y) \in \varepsilon\} \subset \delta$. A map $f: X \rightarrow Y$ is called a *coarse isomorphism* if f is bijective and both maps f and f^{-1} are coarse. In this case the balleans (X, \mathcal{E}_X) and (Y, \mathcal{E}_Y) are called *coarsely isomorphic*. It follows that each ballean (X, \mathcal{E}_X) is coarsely isomorphic to the coarse space $(X, \downarrow \mathcal{E}_X)$.

Coarse isomorphisms play the role of isomorphisms in the coarse category (whose objects are balleans and morphisms are coarse maps). Probably a more important notion is that of a coarse equivalence of balleans. Two balleans (X, \mathcal{E}_X) and (Y, \mathcal{E}_Y) are *coarsely equivalent* if they contain coarsely isomorphic large subspaces $L_X \subset X$ and $L_Y \subset Y$. A subset L of a ballean (X, \mathcal{E}_X) is called *large* if $X = B(L, \varepsilon)$ for some entourage $\varepsilon \in \mathcal{E}_X$.

Coarse equivalences can be alternatively defined using multi-valued maps. By a multi-valued map (briefly, a *multi-map*) $\Phi: X \multimap Y$ between two sets X, Y we understand any subset $\Phi \subset X \times Y$. For a subset $A \subset X$ by $\Phi(A) = \{y \in Y : \exists a \in A \text{ with } (a, y) \in \Phi\}$ we denote the image of A under the multi-map Φ . Given a point $x \in X$ we write $\Phi(x)$ instead of $\Phi(\{x\})$.

The inverse $\Phi^{-1}: Y \multimap X$ of the multi-map $\Phi: X \multimap Y$ is the multi-map

$$\Phi^{-1} = \{(y, x) \in Y \times X : (x, y) \in \Phi\} \subset Y \times X$$

assigning to each point $y \in Y$ the set $\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}$. For two multi-maps $\Phi: X \multimap Y$ and $\Psi: Y \multimap Z$ we define their composition $\Psi \circ \Phi: X \multimap Z$ as usual:

$$\Psi \circ \Phi = \{(x, z) \in X \times Z : \exists y \in Y \text{ such that } (x, y) \in \Phi \text{ and } (y, z) \in \Psi\}.$$

A multi-map $\Phi: X \multimap Y$ between two balleans (X, \mathcal{E}_X) and (Y, \mathcal{E}_Y) is called *coarse* if for every $\varepsilon \in \mathcal{E}_X$ there is an entourage $\delta \in \mathcal{E}_Y$ containing the set $\omega_\Phi(\varepsilon) = \bigcup_{(x, y) \in \varepsilon} \Phi(x) \times \Phi(y)$ called the ε -oscillation of Φ . More precisely, for a function $\varphi: \mathcal{E}_X \rightarrow \mathcal{E}_Y$ a multi-map $\Phi: X \multimap Y$ is defined to be φ -coarse if $\omega_\Phi(\varepsilon) \subset \varphi(\varepsilon)$ for every $\varepsilon \in \mathcal{E}_X$. So, a multi-map $\Phi: X \multimap Y$ is coarse if and only if Φ is φ -coarse for some $\varphi: \mathcal{E}_X \rightarrow \mathcal{E}_Y$. It follows that a (single-valued) map $f: X \rightarrow Y$ is coarse if and only if it is coarse as a multi-map.

A multi-map $\Phi: X \multimap Y$ between two balleans is called a *coarse embedding* if $\Phi^{-1}(Y) = X$ and both maps Φ and Φ^{-1} are coarse. If, in addition, $\Phi(X) = Y$, then the multi-map $\Phi: X \multimap Y$ is called a *coarse equivalence* between the balleans X and Y . By analogy with the proof of Proposition 2.1 [3], it can be shown that two balleans X, Y are coarsely equivalent if and only if there is a coarse equivalence $\Phi: X \multimap Y$.

The study of balleans (or coarse spaces) up to their coarse equivalence is one of principal tasks of Coarse Geometry [5], [6], [13], [14].

Example 1.7. Let G be a group. An ideal \mathcal{I} in the Boolean algebra of all subsets of G is called a *group ideal* if $G = \bigcup \mathcal{I}$ and if for any $A, B \in \mathcal{I}$ we get $AB^{-1} \in \mathcal{I}$.

Let \mathcal{I} be a group ideal \mathcal{I} on a group G and X be a transitive G -space endowed with an action $G \times X \rightarrow X$ of the group G . The G -space X carries the ballean structure $\mathcal{E}_{X, G, \mathcal{I}} = \{\varepsilon_A\}_{A \in \mathcal{I}}$ consisting of the entourages $\varepsilon_A = \{(x, y) \in X : x \in (A \cup \{1_G\} \cup A^{-1}) \cdot y\}$ parametrized by sets $A \in \mathcal{I}$. Here by 1_G we denote the unit of the group G .

By Theorems 1 and 3 of [7], every (cellular) ballean (X, \mathcal{E}_X) is coarsely isomorphic to the ballean $(X, \mathcal{E}_{X, G, \mathcal{I}})$ for a suitable group G of permutations of X and a suitable group ideal \mathcal{I} of G (having a base consisting of subgroups of G).

Example 1.8. Let G be a group endowed with the ballean \mathcal{E}_G consisting of entourages $\varepsilon_F = \{(x, y) \in G \times G : xy^{-1} \in F\}$ parametrized by finite subsets $F = F^{-1} \subset G$ containing the unit 1_G of the group. By [11, 9.8] the ballean (G, \mathcal{E}_G) is cellular if and only if the group G is locally finite (in the sense that each finite subset of G is contained in a finite subgroup of G). By [3], any two infinite countable locally finite groups G, H are coarsely equivalent. On the other hand, by [9], two countable locally finite groups G, H are coarsely isomorphic if and

only if $\phi_G = \phi_H$. Here for a group G by $\phi_G: \Pi \rightarrow \omega \cup \{\omega\}$ we denote its *factorizing function*. It is defined on the set Π of prime numbers and assigns to each prime number $p \in \Pi$ the (finite or infinite) number

$$\phi_G(p) = \sup\{k \in \omega : G \text{ contains a subgroup of cardinality } p^k\}.$$

2. A CRITERION FOR A COARSE EQUIVALENCE OF TWO CELLULAR ORDINAL BALLEANS

In this section we introduce two cardinal characteristics called covering numbers of a ballean, and using these cardinal characteristics give a criterion for a coarse equivalence of two cellular ordinal balleanes.

Given a subset $A \subset X$ of a set X and an entourage $\varepsilon \in X \times X$ consider the cardinal

$$\text{cov}_\varepsilon(A) = \min\{|C| : C \subset X, A \subset B(C, \varepsilon)\}$$

equal to the smallest number of ε -balls covering the set A .

For every ballean (X, \mathcal{E}_X) consider the following cardinals:

- $\text{cov}^\#(X, \mathcal{E}_X)$, equal to the smallest cardinal κ for which there is an entourage $\varepsilon \in \mathcal{E}_X$ such that for every $\delta \in \mathcal{E}_X$ we get $\sup_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) < \kappa$;
- $\text{cov}^b(X, \mathcal{E}_X)$, equal to the largest cardinal κ such that for any cardinal $\lambda < \kappa$ and entourage $\varepsilon \in \mathcal{E}_X$ there is $\delta \in \mathcal{E}_X$ such that $\min_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) \geq \lambda$.

It follows that

$$\text{cov}^\#(X) = \min_{\varepsilon \in \mathcal{E}_X} \sup_{\delta \in \mathcal{E}_X} \left(\sup_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) \right)^+$$

and

$$\text{cov}^b(X) = \min_{\varepsilon \in \mathcal{E}_X} \sup_{\delta \in \mathcal{E}_X} \left(\min_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) \right)^+,$$

where κ^+ denotes the smallest cardinal which is larger than κ . Cardinals are identified with the smallest ordinals of given cardinality.

The following proposition can be proved by analogy with the proof of Lemmas 3.1 and 3.2 in [1].

Proposition 2.1. *If a ballean X coarsely embeds into a ballean Y , then $\text{cov}^\#(X) \leq \text{cov}^\#(Y)$. If balleanes X, Y are coarsely equivalent, then $\text{cov}^b(X) = \text{cov}^b(Y)$ and $\text{cov}^\#(X) = \text{cov}^\#(Y)$.*

Observe that the inequality $\text{cov}^\#(X) \leq \omega$ means that X has bounded geometry while $\text{cov}^b(X) \geq \omega$ means that X has no isolated balls (see [2]). By [3], any two metrizable cellular balleanes of bounded geometry and without isolated balls are coarsely equivalent. In [1] this result was extended to the following criterion: two metrizable cellular balleanes X, Y are coarsely equivalent if $\text{cov}^b(X) = \text{cov}^\#(X) = \text{cov}^\#(Y) = \text{cov}^b(Y)$. In this paper we further extend this criterion to cellular ordinal balleanes and prove the following main result of this paper.

Theorem 2.2. *Let X, Y be any two cellular ordinal balleanes with $\text{cof}(X) = \text{cof}(Y)$.*

- (1) *If $\text{cov}^\#(X) \leq \text{cov}^b(Y)$, then X is coarsely equivalent to a subspace of Y .*
- (2) *If $\text{cov}^b(X) = \text{cov}^\#(X) = \text{cov}^\#(Y) = \text{cov}^b(Y)$, then the balleanes X and Y are coarsely equivalent.*

The proof of this theorem will be presented in Section 6. First we shall discuss some applications of this theorem.

3. CLASSIFYING HOMOGENEOUS CELLULAR ORDINAL BALLEANS

In this section we shall apply Theorem 2.2 to show that for a cellular ordinal ballean X the equality $\text{cov}^b(X) = \text{cov}^\#(X)$ is equivalent to the homogeneity of X , defined as follows.

A ballean (X, \mathcal{E}_X) is called *homogeneous* if there is a function $\varphi: \mathcal{E}_X \rightarrow \mathcal{E}_X$ such that for every points $x, y \in X$ there is a coarse equivalence $\Phi: X \rightarrow X$ such that $y \in \Phi(x)$ and the multi-maps Φ and Φ^{-1} are φ -coarse. Let us recall that $\Phi: X \rightarrow X$ is φ -coarse if $\omega_\Phi(\varepsilon) := \bigcup_{(x, y) \in \varepsilon} \Phi(x) \times \Phi(y) \subset \varphi(\varepsilon)$ for every $\varepsilon \in \mathcal{E}_X$.

The following proposition shows that homogeneity is preserved by coarse equivalences.

Proposition 3.1. *A ballean X is homogeneous if and only if it is coarsely equivalent to a homogeneous ballean Y .*

Proof. The “only if” part is trivial. To prove the “if” part, assume that a ballean (X, \mathcal{E}_X) admits a coarse equivalence $\Phi: X \rightarrow Y$ with a homogeneous ballean (Y, \mathcal{E}_Y) . By the homogeneity of (Y, \mathcal{E}_Y) , there is a function $\varphi_Y: \mathcal{E}_Y \rightarrow \mathcal{E}_Y$ such that for any points $y, y' \in Y$ there is a coarse equivalence $\Psi: Y \rightarrow Y$ such that $y' \in \Psi(y)$ and both multi-maps Ψ and Ψ^{-1} are φ_Y -coarse. Since Φ is a coarse equivalence, there are functions

$\varphi_{X,Y} : \mathcal{E}_X \rightarrow \mathcal{E}_Y$ and $\varphi_{Y,X} : \mathcal{E}_Y \rightarrow \mathcal{E}_X$ such that $\omega_\Phi(\varepsilon) \subset \varphi_{X,Y}(\varepsilon)$ and $\omega_{\Phi^{-1}}(\delta) \subset \varphi_{Y,X}(\delta)$ for every $\varepsilon \in \mathcal{E}_X$ and $\delta \in \mathcal{E}_Y$. We claim that the function

$$\varphi_X = \varphi_{Y,X} \circ \varphi_Y \circ \varphi_{X,Y} : \mathcal{E}_X \rightarrow \mathcal{E}_X$$

witnesses that the ballean X is homogeneous. Indeed, given any points x, x' , we can choose points $y \in \Phi(x)$, $y' \in \Phi(x')$ and find a coarse equivalence $\Psi_Y : Y \rightarrow Y$ such that $y' \in \Psi_Y(y)$ and the multi-maps Ψ_Y and Ψ_Y^{-1} are φ_Y -coarse. It can be shown that the multi-map $\Psi_X = \Phi^{-1} \circ \Psi_Y \circ \Phi : X \rightarrow X$ has the desired properties: $x' \in \Phi^{-1}(y') \subset \Phi^{-1}(\Psi_Y(y)) \subset \Phi^{-1}(\Psi_Y(\Phi(x))) = \Phi_X(x)$ and $\omega_{\Phi_X}(\varepsilon) \cup \omega_{\Phi_X^{-1}}(\varepsilon) \subset \varphi_X(\varepsilon)$ for all $\varepsilon \in \mathcal{E}_X$. \square

Proposition 3.2. *If a ballean X is homogeneous, then $\text{cov}^b(X) = \text{cov}^\sharp(X)$.*

Proof. Since $\text{cov}^b(X) \leq \text{cov}^\sharp(X)$, it suffices to check that $\text{cov}^b(X) \geq \text{cov}^\sharp(X)$. This inequality will follow as soon as given $\varepsilon \in \mathcal{E}_X$ and a cardinal $\kappa < \text{cov}^\sharp(X)$, we find an entourage $\delta \in \mathcal{E}_X$ such that $\min_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) \geq \kappa$. By the homogeneity of X , there is a function $\varphi : \mathcal{E}_X \rightarrow \mathcal{E}_X$ such that for any points $x, y \in X$ there is a coarse equivalence $\Phi : X \rightarrow X$ such that $y \in \Phi(x)$ and the multi-maps Φ and Φ^{-1} are φ -coarse.

By the definition of the cardinal $\text{cov}^\sharp(X) > \kappa$, for the entourage $\varepsilon' = \varphi(\varepsilon)$, there is an entourage $\delta' \in \mathcal{E}_X$ and a point $x' \in X$ such that $\text{cov}_{\varepsilon'}(B(x', \delta')) \geq \kappa$. We claim that the entourage $\delta = \varphi(\delta') \in \mathcal{E}_X$ has the required property: $\min_{x \in X} \text{cov}_\varepsilon(B(x, \delta)) \geq \kappa$. Assume conversely that $\text{cov}_\varepsilon(B(x, \delta)) < \kappa$ for some $x \in X$. By the homogeneity of X and the choice of φ , there is a coarse equivalence $\Phi : X \rightarrow X$ such that $x' \in \Phi(x)$ and the multi-maps Φ and Φ^{-1} are φ -coarse. Since $\text{cov}_\varepsilon(B(x, \delta)) < \kappa$, there is a subset $C \subset X$ of cardinality $|C| < \kappa$ such that $B(x, \delta) \subset \bigcup_{c \in C} B(c, \varepsilon)$. For every $c \in C$ fix a point $y_c \in \Phi(c)$ and observe that for every point $b \in B(c, \varepsilon)$ we get $(b, c) \in \varepsilon$ and hence $\Phi(b) \times \Phi(c) \subset \omega_\Phi(\varepsilon) \subset \varphi(\varepsilon)$ and $\Phi(b) \subset B(y_c, \varphi(\varepsilon)) = B(y_c, \varepsilon')$, which implies $\Phi(B(c, \varepsilon)) \subset B(y_c, \varepsilon')$. Taking into account that $B(x, \delta) \subset \bigcup_{c \in C} B(c, \varepsilon)$, we get $\Phi(B(x, \delta)) \subset \bigcup_{c \in C} \Phi(B(c, \varepsilon)) \subset \bigcup_{c \in C} B(y_c, \varepsilon')$, which implies $\text{cov}_{\varepsilon'}(\Phi(B(x, \delta))) \leq |C| < \kappa$.

We claim that $B(x', \delta') \subset \Phi(B(x, \delta))$. Indeed, for any point $y' \in B(x', \delta')$ we can fix a point $y \in \Phi^{-1}(x')$ and observe that $(y', x') \in \delta'$ implies $(y, x) \in \Phi^{-1}(y') \times \Phi^{-1}(x') \subset \omega_{\Phi^{-1}}(\delta') \subset \varphi(\delta') = \delta$ (by the φ -coarse property of the multi map Φ^{-1}). Then $y \in B(x, \delta)$ and $y' \in \Phi(y) \subset \Phi(B(x, \delta))$. Finally, we get $B(x', \delta') \subset \Phi(B(x, \delta))$ and $\text{cov}_{\varepsilon'}(B(x', \delta')) \leq \text{cov}_{\varepsilon'}(\Phi(B(x, \delta))) \leq |C| < \kappa$, which contradicts the choice of δ' and x' . This completes the proof of the equality $\text{cov}^b(X) = \text{cov}^\sharp(X)$. \square

Theorem 3.3. *A cellular ordinal ballean X is homogeneous if and only if $\text{cov}^b(X) = \text{cov}^\sharp(X)$.*

Proof. The “only if” part follows from Proposition 3.2. To prove the “if” part, assume X is a cellular ordinal ballean with $\text{cov}^b(X) = \text{cov}^\sharp(X)$. Let $\gamma = \text{cof}(X) = \text{add}(X)$. The definition of the cardinal $\kappa = \text{cov}^b(X) = \text{cov}^\sharp(X)$ implies that there exists a non-decreasing transfinite sequence of cardinals $(\kappa_\alpha)_{\alpha < \gamma}$ such that $\kappa = \sup_{\alpha < \gamma} \kappa_\alpha^+$. Choose an increasing transfinite sequence of groups $(G_\alpha)_{\alpha < \gamma}$ such that $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$ for every limit ordinal $\alpha < \gamma$ and $|G_{\alpha+1}/G_\alpha| = \kappa_\alpha$ for every ordinal $\alpha < \gamma$.

Consider the group $G = \bigcup_{\alpha < \gamma} G_\alpha$ endowed with the ballean structure $\mathcal{E}_G = (\varepsilon_\alpha)_{\alpha < \gamma}$ consisting of the entourages

$$\varepsilon_\alpha = \{(x, y) \in G : x^{-1}y \in G_\alpha\} \text{ for } \alpha < \gamma.$$

It is clear that the left shifts are id-coarse isomorphisms of (G, \mathcal{E}_G) , which implies that the ballean (G, \mathcal{E}_G) is homogeneous. It is clear that $\text{add}(G, \mathcal{E}_G) = \text{cof}(G, \mathcal{E}_G) = \gamma$ and

$$\text{cov}^b(G, \mathcal{E}_G) = \text{cov}^\sharp(G, \mathcal{E}_G) = \min_{\alpha < \gamma} \sup_{\alpha \leq \beta < \gamma} |G_\beta/G_\alpha|^+ = \sup_{\alpha < \gamma} \kappa_\alpha^+ = \kappa.$$

Applying Theorem 2.2, we conclude that X is coarsely equivalent to the homogeneous ballean (G, \mathcal{E}_G) and hence X is homogeneous according to Proposition 3.1. \square

The following corollary of Theorems 2.2 and 3.3 shows that the cardinals $\text{cof}(X)$ and $\text{cov}^\sharp(X)$ fully determine the coarse structure of a homogeneous cellular ordinal ballean X .

Theorem 3.4. *For any two homogeneous cellular ordinal balleans X, Y the following conditions are equivalent:*

- (1) X and Y are coarsely equivalent;
- (2) X is coarsely equivalent to a subspace of Y and Y is coarsely equivalent to a subspace of X ;
- (3) $\text{cof}(X) = \text{cof}(Y)$ and $\text{cov}^\sharp(X) = \text{cov}^\sharp(Y)$.

Proof. The implication (1) \Rightarrow (2) is trivial, the implication (2) \Rightarrow (3) follows by the invariance of the cardinal characteristics cof and cov^\sharp under coarse equivalence and their monotonicity under taking subspaces, and the final implication (3) \Rightarrow (1) follows from Theorems 2.2 and 3.3. \square

4. RECOGNIZING THE COARSE STRUCTURE OF CANTOR MACRO-CUBES AND CARDINAL BALLEANS

It is easy to see that for any ordinal γ and transfinite sequence $(\kappa_\alpha)_{\alpha \in \gamma}$ of non-zero cardinals the asymptotic product $\prod_{\alpha \in \gamma} \kappa_\alpha$ is a homogeneous cellular ordinal ballean whose cofinality equals $\text{cf}(\gamma)$, the cofinality of the ordinal γ . In particular, the Cantor macro-cube $2^{<\gamma}$ is a homogeneous cellular ordinal ballean of cofinality $\text{cof}(2^{<\gamma}) = \text{cf}(\gamma)$.

To evaluate the covering numbers of $2^{<\gamma}$, for an ordinal γ , consider the ordinal

$$\lfloor \gamma \rfloor = \min\{\alpha : \gamma = \beta + \alpha \text{ for some } \beta < \gamma\}$$

called the *tail* of γ , and the cardinal

$$\lceil \gamma \rceil = \min\{\alpha : \gamma \leq \beta + |\alpha| \text{ for some } \beta < \gamma\}$$

called the *cardinal tail* of γ . It is clear that $\lfloor \gamma \rfloor \leq \lceil \gamma \rceil$. Moreover,

$$\lceil \gamma \rceil = \begin{cases} |\lfloor \gamma \rfloor| & \text{if } \lfloor \gamma \rfloor \text{ is a cardinal,} \\ |\lfloor \gamma \rfloor|^+ & \text{otherwise.} \end{cases}$$

The equality $\gamma = \lfloor \gamma \rfloor$ holds if and only if the ordinal γ is *additively indecomposable*, which means that $\alpha + \beta < \gamma$ for any ordinals $\alpha, \beta < \gamma$.

The following proposition can be derived from the definition of $2^{<\gamma}$.

Proposition 4.1. *For every ordinal γ the Cantor macro-cube $2^{<\gamma}$ is a cellular ordinal ballean with*

$$\text{add}(2^{<\gamma}) = \text{cof}(2^{<\gamma}) = \text{cf}(\gamma) \quad \text{and} \quad \text{cov}^b(2^{<\gamma}) = \text{cov}^\#(2^{<\gamma}) = \lceil \gamma \rceil.$$

The following theorem (which can be derived from Proposition 4.1 and Theorem 2.2) shows that in the class of cellular ordinal balleans, the Cantor macro-cubes $2^{<\gamma}$ play a role analogous to the role of the Cantor cubes 2^κ in the class of zero-dimensional compact Hausdorff spaces.

Theorem 4.2. *Let γ be any ordinal and X be any cellular ordinal ballean such that $\text{cof}(X) = \text{cf}(\gamma)$.*

- (1) *If $\lceil \gamma \rceil \leq \text{cov}^b(X)$, then $2^{<\gamma}$ is coarsely equivalent to a subspace of X ;*
- (2) *If $\text{cov}^\#(X) \leq \lceil \gamma \rceil$, then X is coarsely equivalent to a subspace of $2^{<\gamma}$.*
- (3) *If $\text{cov}^b(X) = \text{cov}^\#(X) = \lceil \gamma \rceil$, then X is coarsely equivalent to $2^{<\gamma}$.*

Proposition 4.1 and Theorem 4.2 imply the following characterization of the Cantor macro-cube $2^{<\gamma}$ which extends the characterization of the Cantor macro-cube $2^{<\omega}$ proved in [3].

Theorem 4.3. *For any ordinal γ and any ballean X the following conditions are equivalent:*

- (1) *X is coarsely equivalent to $2^{<\gamma}$;*
- (2) *X is cellular, $\text{add}(X) = \text{cof}(X) = \text{cf}(\gamma)$ and $\text{cov}^b(X) = \text{cov}^\#(X) = \lceil \gamma \rceil$.*

Corollary 4.4. *For any two ordinals β, γ the Cantor macro-cubes $2^{<\beta}$ and $2^{<\gamma}$ are coarsely equivalent if and only if $\text{cf}(\beta) = \text{cf}(\gamma)$ and $\lceil \beta \rceil = \lceil \gamma \rceil$.*

Finally, we recognize the coarse structure on the ballean $\overleftrightarrow{\gamma}$ supported by an additively indecomposable ordinal γ . Given any non-zero ordinal γ we consider the family $\{\varepsilon_\alpha\}_{\alpha < \gamma}$ of the entourages

$$\varepsilon_\alpha = \{(x, y) \in \gamma \times \gamma : x \leq y + \alpha \text{ and } y \leq x + \alpha\}$$

for $\alpha < \gamma$. It is easy to see that $(\gamma, \{\varepsilon_\alpha\}_{\alpha < \gamma})$ is a ballean if and only if the ordinal γ is additively indecomposable (which means that $\alpha + \beta < \gamma$ for any ordinals $\alpha, \beta < \gamma$). This ballean will be denoted by $\overleftrightarrow{\gamma}$.

The following theorem classifies the balleans $\overleftrightarrow{\gamma}$ up to coarse equivalence.

Theorem 4.5. *For any additively indecomposable ordinal γ the ballean $\overleftrightarrow{\gamma}$ is coarsely equivalent to:*

- $\overleftrightarrow{\omega}$ if and only if $\gamma = \beta \cdot \omega$ for some β ;
- $2^{<\gamma}$, otherwise.

Proof. If $\gamma = \beta \cdot \omega$ for some ordinal β , then the ballean $\overleftrightarrow{\gamma}$ is coarsely equivalent to $\overleftrightarrow{\omega}$ since $\overleftrightarrow{\gamma}$ contains the large subset $L = \{\beta \cdot n : n \in \omega\}$, which is coarsely isomorphic to $\overleftrightarrow{\omega}$.

Now assume that $\gamma \neq \beta \cdot \omega$ for any ordinal β . Since γ is additively indecomposable, this means that $\beta \cdot \omega < \gamma$ for any ordinal $\beta < \gamma$, which implies that the ballean $\overleftrightarrow{\gamma}$ is cellular. Since $\text{add}(\overleftrightarrow{\gamma}) = \text{cof}(\overleftrightarrow{\gamma}) = \text{cf}(\gamma)$ and $\text{cov}^b(\overleftrightarrow{\gamma}) = \text{cov}^\#(\overleftrightarrow{\gamma}) = \lceil \gamma \rceil$, the cellular ordinal ballean $\overleftrightarrow{\gamma}$ is coarsely equivalent to $2^{<\gamma}$ according to Theorem 4.3. \square

Remark 4.6. Let us observe that for any ordinal γ the balleans $2^{<\gamma}$ and $\overleftrightarrow{\omega}$ are not coarsely equivalent since the ballean $2^{<\gamma}$ is cellular whereas $\overleftrightarrow{\omega}$ is not.

5. EMBEDDING CELLULAR ORDINAL BALLEANS INTO ASYMPTOTIC PRODUCTS OF CARDINALS

In this section we construct coarse embeddings of cellular ordinal balleans into asymptotic products of cardinals. This embedding will play a crucial role in the proof of Theorem 2.2 presented in the next section.

Let us observe that for any transfinite sequence of cardinals $(\kappa_\alpha)_{\alpha < \gamma}$ the asymptotic product $\prod_{\alpha < \gamma} \kappa_\alpha$ carries an operation of coordinatewise addition of sequences induces by the operation of addition of ordinals. For ordinals $\beta < \gamma$ and $y \in \kappa_\alpha$ let $y \cdot \delta_\beta$ denote the sequence $(x_\alpha)_{\alpha < \gamma} \in \prod_{\alpha < \gamma} \kappa_\alpha$ such that $x_\alpha = y$ if $\alpha = \beta$ and $x_\alpha = 0$ otherwise. It follows that each element $(x_\alpha)_{\alpha < \gamma} \in \prod_{\alpha < \gamma} \kappa_\alpha$ can be written as $\sum_{\alpha \in A} x_\alpha \cdot \delta_\alpha$ for the finite set $A = \{\alpha < \gamma : x_\alpha \neq 0\}$.

The following lemma exploits and develops the decomposition technique used in [9], [11, §10] and [12].

Lemma 5.1. *Let X be an ordinal ballean of infinite cofinality γ and $(\varepsilon_\alpha)_{\alpha < \gamma}$ be a well-ordered base of the coarse structure of X consisting of cellular entourages such that $\varepsilon_\beta = \bigcup_{\alpha < \beta} \varepsilon_\alpha$ for all limit ordinals $\beta < \gamma$. For every $\alpha < \gamma$ and $x \in X$ let $\kappa_\alpha(x) = \text{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1}))$ and put $\kappa_\alpha = \min_{x \in X} \kappa_\alpha(x)$ and $\bar{\kappa}_\alpha = \sup_{x \in X} \kappa_\alpha(x)$. Then the ballean X is coarsely equivalent to a subballean $Y \subset \prod_{\alpha < \gamma} \bar{\kappa}_\alpha$ containing the set $\prod_{\alpha < \gamma} \kappa_\alpha$.*

Proof. For each two points $x, y \in X$ let

$$d(x, y) = \min\{\alpha < \gamma : (x, y) \in \varepsilon_\alpha\}$$

and observe that for any pair $(x, y) \notin \varepsilon_0$ the ordinal $d(x, y)$ is not limit (as $\varepsilon_\beta = \bigcup_{\alpha < \beta} \varepsilon_\alpha$ for any limit ordinal $\beta < \gamma$). Consequently we can find an ordinal $d^-(x, y)$ such that $d(x, y) = d^-(x, y) + 1$.

Fix any well-ordering \preceq of the set X . Given a non-empty subset $B \subset X$ denote by $\min B$ the smallest point of B with respect to the well-order \preceq and for every $\alpha < \gamma$ let $c_\alpha : X \rightarrow X$ be the map assigning to each point $x \in X$ the smallest element $c_\alpha(x) = \min B(x, \varepsilon_\alpha)$ of the ball $B(x, \varepsilon_\alpha)$. Since ε_α is an equivalence relation, $B(x, \varepsilon_\alpha) = B(c_\alpha(x), \varepsilon_\alpha)$. To simplify the notation in the sequel we shall denote the ball $B(x, \varepsilon_\alpha)$ by $B_\alpha(x)$.

Observe that for every $\alpha < \gamma$ and ball $B \in \{B_{\alpha+1}(x) : x \in X\}$ the set $c_\alpha(X) \cap B$ has cardinality $\kappa_\alpha(\min B)$, so we can fix a map $n_{\alpha, B} : B \rightarrow \kappa_\alpha(\min B)$ such that $\{B_\alpha(x) : x \in B\} = \{n_{\alpha, B}^{-1}(\beta) : \beta \in \kappa_\alpha(\min B)\}$ and $n_{\alpha, B}^{-1}(0) = B_\alpha(\min B)$. Finally, define a map $n_\alpha : X \rightarrow \bar{\kappa}_\alpha$ assigning to each point $y \in X$ the number $n_\alpha(y) := n_{\alpha, B_{\alpha+1}(y)}(y)$ of the ε_α -ball containing y in the partition of the $\varepsilon_{\alpha+1}$ -ball $B_{\alpha+1}(y)$. The definition of the cardinal κ_α implies that $\kappa_\alpha \subset \kappa_\alpha(x) = n_\alpha(B_{\alpha+1}(x))$ for every $x \in X$.

For every $x \in X$ define a map $f_x : X \rightarrow \prod_{\alpha < \gamma} \bar{\kappa}_\alpha$ by the recursive formula

$$f_x(y) = \begin{cases} 0 & \text{if } d(x, y) = 0; \\ f_{\min B_{d^-(x, y)}(y)}(y) + n_{d^-(x, y)}(y) \cdot \delta_{d^-(x, y)} & \text{otherwise.} \end{cases}$$

Since $d(\min B_{d^-(x, y)}(y), y) < d(x, y)$ the function f_x is well-defined.

It can be shown that for every $x \in X$ the function $f_x : X \rightarrow \prod_{\alpha < \gamma} \bar{\kappa}_\alpha$ determines a coarse equivalence of X with the subspace $f(X)$ of $\prod_{\alpha < \gamma} \bar{\kappa}_\alpha$ containing the asymptotic product $\prod_{\alpha < \gamma} \kappa_\alpha$. \square

6. PROOF OF THEOREM 2.2

Assume that X, Y are two cellular balleans with $\gamma = \text{add}(X) = \text{cof}(X) = \text{cof}(Y) = \text{add}(Y)$ and $\kappa = \text{cov}^b(X) = \text{cov}^\#(X) = \text{cov}^\#(Y) = \text{cov}^b(Y)$ for some cardinals γ and κ . Let $\mathcal{E}_X, \mathcal{E}_Y$ denote the ballean structures of X and Y , respectively.

Separately we shall consider 4 cases.

- 1) $\gamma = 0$. In this case the balleans X, Y are empty and hence coarsely equivalent.
- 2) $\gamma = 1$. In this case the balleans X, Y are bounded and hence are coarsely equivalent.
- 3) $\gamma = \omega$. Since X is a cellular ballean with $\text{cof}(X) = \gamma = \omega$, the coarse structure $\downarrow \mathcal{E}_X$ of X has a well-ordered base $\{\varepsilon_n\}_{n \in \omega}$ consisting of equivalence relations such that $\varepsilon_0 = \Delta_X$. In this case the formula

$$d_X(x, x') = \min\{n \in \omega : (x, x') \in \varepsilon_n\}$$

defines an ultrametric $d_X : X \times X \rightarrow \omega$ generating the coarse structure of the ballean X . By analogy we can define an ultrametric d_Y generating the coarse structure of the ballean Y . Since $\text{cov}^b(X) = \text{cov}^\#(X) = \text{cov}^\#(Y) = \text{cov}^b(Y)$, we can apply Theorem 1.2 of [1] (proved by the technique of towers created in [3]) to conclude that the ultrametric spaces X and Y are coarsely equivalent.

- 4) $\gamma > \omega$. Since X, Y are ordinal balleans of cofinality $\text{cof}(X) = \text{cof}(Y) = \gamma$, we can fix well-ordered bases $\{\tilde{\varepsilon}_\alpha\}_{\alpha < \gamma}$ and $\{\tilde{\delta}_\alpha\}_{\alpha < \gamma}$ of the coarse structures $\downarrow \mathcal{E}_X$ and $\downarrow \mathcal{E}_Y$, respectively.

By induction on $\alpha < \gamma$ we shall construct well-ordered sequences $\{\varepsilon_\alpha\}_{\alpha < \gamma} \subset \downarrow \mathcal{E}_X$ and $\{\delta_\alpha\}_{\alpha < \gamma} \subset \downarrow \mathcal{E}_Y$ such that for every $\alpha < \gamma$ the following conditions will be satisfied:

- (a) $\varepsilon_\alpha = \bigcup_{\beta < \alpha} \varepsilon_\beta$ and $\delta_\alpha = \bigcup_{\beta < \alpha} \delta_\beta$ if the ordinal α is limit;
- (b) ε_α and δ_α are cellular entourages;
- (c) $\tilde{\varepsilon}_\alpha \subset \varepsilon_{\alpha+1}$ and $\tilde{\delta}_\alpha \subset \delta_{\alpha+1}$;
- (d) $\min_{x \in X} \text{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1})) = \sup_{x \in X} \text{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1})) = \min_{y \in Y} \text{cov}_{\delta_\alpha}(B(y, \delta_{\alpha+1})) = \sup_{y \in Y} \text{cov}_{\delta_\alpha}(B(y, \delta_{\alpha+1})) = \kappa_\alpha$ for some cardinal κ_α .

We start the inductive construction by choosing cellular entourages $\varepsilon_0 \in \mathcal{E}_X$ and $\delta_0 \in \mathcal{E}_Y$ such that

$$\sup_{x \in X} \text{cov}_{\varepsilon_0}(B(x, \varepsilon_0)) < \kappa \quad \text{and} \quad \sup_{y \in Y} \text{cov}_{\delta_0}(B(y, \delta_0)) < \kappa$$

for any entourages $\varepsilon \in \downarrow \mathcal{E}_X$ and $\delta \in \downarrow \mathcal{E}_Y$. The existence of such entourages ε_0 and δ_0 follows from the cellularity of X, Y and the definition of the cardinals $\text{cov}^\#(X) = \text{cov}^\#(Y) = \kappa$. Assume that for some ordinal $\alpha < \gamma$ and all ordinals $\beta < \alpha$ the cellular entourages ε_β and δ_β have already been constructed. If the ordinal α is limit, then we put $\varepsilon_\alpha = \bigcup_{\beta < \alpha} \varepsilon_\beta$ and $\delta_\alpha = \bigcup_{\beta < \alpha} \delta_\beta$ and observe that the entourages ε_α and δ_α are cellular as unions of increasing chains of cellular entourages. Moreover, $\varepsilon_\alpha \in \downarrow \mathcal{E}_X$ and $\delta_\alpha \in \downarrow \mathcal{E}_Y$ as $\alpha < \gamma = \text{add}(X) = \text{add}(Y)$.

Next, assume that the ordinal α is not limit and hence $\alpha = \beta + 1$ for some ordinal β . Taking into account the choice of the entourages ε_0, δ_0 and using the definitions of the cardinals $\text{cov}^b(X) = \text{cov}^b(Y)$, we can construct two increasing sequences of cellular entourages $\{\varepsilon'_n\}_{n \in \omega} \subset \downarrow \mathcal{E}_X$ and $\{\delta'_n\}_{n \in \omega} \subset \downarrow \mathcal{E}_Y$ such that

$$\sup_{x \in X} \text{cov}_{\varepsilon'_n}(B(x, \varepsilon'_{n+1})) \leq \min_{y \in Y} \text{cov}_{\delta'_n}(B(y, \delta'_{n+1}))$$

and

$$\sup_{y \in Y} \text{cov}_{\delta'_n}(B(y, \delta'_{n+1})) \leq \min_{x \in X} \text{cov}_{\varepsilon'_{n+1}}(B(x, \varepsilon'_{n+2})).$$

The entourages ε'_1 and δ'_1 can be chosen so that $\tilde{\varepsilon}_\alpha \subset \varepsilon'_1$ and $\tilde{\delta}_\alpha \subset \delta'_1$. Since $\text{add}(X) = \text{add}(Y) > \omega$, the entourages $\varepsilon_{\alpha+1} = \bigcup_{n \in \omega} \varepsilon'_n$ and $\delta_{\alpha+1} = \bigcup_{n \in \omega} \delta'_n$ belong to the coarse structures $\downarrow \mathcal{E}_X$ and $\downarrow \mathcal{E}_Y$, respectively, and have the properties (b)–(d), required in the inductive construction.

By Lemma 5.1, there are coarse equivalences $f_X: X \rightarrow \prod_{\alpha < \gamma} \kappa_\alpha$ and $f_Y: Y \rightarrow \prod_{\alpha < \gamma} \kappa_\alpha$. Then the multi-map $f_Y^{-1} \circ f_X: X \multimap Y$ is a coarse equivalence between the ballean X and Y . \square

REFERENCES

- [1] T. Banakh, D. Repovš, *Classifying homogeneous ultrametric spaces up to coarse equivalence*, Colloq. Math. (to appear); available at (<http://arxiv.org/abs/1408.4818>).
- [2] T. Banakh, I. Zarichnyi, *A coarse characterization of the Baire macro-space*, Proc. of Intern. Geometry Center. 3:4 (2010) 6–14 (available <http://arxiv.org/abs/1103.5118>).
- [3] T. Banakh, I. Zarichnyi, *Characterizing the Cantor bi-cube in asymptotic categories*, Groups Geom. Dyn. **5**:4 (2011), 691–728.
- [4] N. Brodsky, J. Dydak, J. Higes, A. Mitra, *Dimension zero at all scales*, Topology Appl. **154**:14 (2007), 2729–2740.
- [5] S. Buyalo, V. Schroeder, *Elements of Asymptotic Geometry*, EMS Monographs in Mathematics. (EMS), Zürich, 2007.
- [6] P. Nowak, G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics. (EMS), Zürich, 2012.
- [7] O. Petrenko, I.V. Protasov, *Balleans and G-spaces*, Ukr. Math. J. **64** (2012), 344–350.
- [8] O. Petrenko, I.V. Protasov, S. Slobodianiuk, *Asymptotic structures of cardinals*, Appl. Gen. Topology, **15**:2 (2014), 137–146.
- [9] I.V. Protasov, *Morphisms of balls structures of groups and graphs*, Ukr. Mat. Zh. **54**:6 (2002), 847–855.
- [10] I.V. Protasov, *Normal ball structures*, Math. Stud. **20** (2003), 3–16.
- [11] I. Protasov, T. Banakh, *Ball structures and colorings of graphs and groups*, VNTL Publ. 2003, 148p.
- [12] I.V. Protasov, A. Tsvietkova, *Decomposition of cellular ballean*, Top. Proc. **36** (2010), 77–83.
- [13] I. Protasov M. Zarichnyi, *General Asymptology*, Monograph Series, Vol.12, VNTL Publ., Lviv, 2007.
- [14] J. Roe, *Lectures on Coarse Geometry*, Univ. Lecture Series, Vol.31, Amer. Math. Soc., 2003.

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